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ON THE FREEZING TIME OF CYLINDERS AND SPHERES

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The Stefan problem is considered for freezing of spherical and cylindrical bodies submerged in a cooling agent. A method is suggested for analytical solution of this problem. Formulas are obtained for the freezing time and volume-average temperatures of a cylinder and sphere after completion of the freezing process.

1. Calculation of time necessary for freezing of a product and its subsequent holding at the volume-average temperature necessary for storage in a fast-freezing apparatus is important in food technology.

The expression known in refrigeration technology as the Plank formula is most often used for determination of the freezing time:

$$t_{\mathrm{f}} = \Theta \frac{R\rho q}{T_{\mathrm{cry}} - T_{\mathrm{c.a}}} \left(\frac{R}{2\lambda} + \frac{1}{\alpha}\right) . \tag{1}$$

Formula (1) was obtained with the following simplifying assumptions:

1. Before freezing, the body was cooled to the cryoscopic temperature.
2. In this body ice is formed without supercooling at the cryoscopic temperature.
3. The body is homogeneous: the thermophysical properties of its frozen part are independent of temperature, the heat transfer coefficient and temperature of the cooling agent are independent of time, and the density of the body does not change during freezing.
4. The heat capacity of the frozen part of the body is zero.

There are many modifications of formula (1) that allow assumptions 1–4 to be omitted. They can be divided into empirical, experimental correction factors entering into formula (1), and analytical, of which Ryutov's [1], Geints and Yushkov's [2], and Brazhnikov's works [3] seem to be most interesting. In those works the authors constructed approximate models of the dynamics of the freezing process for the case of a flat plate with assumption 4 rejected. Their general idea can be characterized as follows: it is assumed that there exist phase interfaces between the frozen and unfrozen parts of the plate and that in the freezing process these interfaces move from the surface into the interior of the body. The freezing process is assumed to be completed when these interfaces join in the center of the plate. This assumption is not quite realistic, but it can be adopted as a first approximation. With this assumption, analytical description of the freezing process is reduced to solution of the so-called Stefan problem.

In [1-3], for approximate solution of this problem, the temperature distribution in the frozen part of the plate was approximated by a certain expression that was substituted into equations of the Stefan problem. Then, the time function of the thickness of the frozen layer was calculated and, as a consequence, the freezing time was found.

In the present work a similar calculation procedure is used for a cylinder and sphere. It is most similar to the procedure used by the authors of [2], where for a plate with thickness $2R$ the following expression was obtained for the freezing time:

$$t_{\mathrm{f}} = \frac{R\rho q}{T_{\mathrm{cry}} - T_{\mathrm{c.a}}} \left(\frac{R}{2\lambda} + \frac{1}{\alpha}\right) + \frac{R\rho C}{2} \left\{\frac{R}{2\lambda} + \frac{1}{\alpha}\left(1 - \frac{\ln(1 + \mathrm{Bi})}{\mathrm{Bi}}\right)\right\} . \tag{2}$$

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After completion of freezing, the temperature distribution in the plate appears linear:

$$T(x;\, t_f) = T_{cry} - \frac{\text{Bi}}{\text{Bi} + 1}(T_{cry} - T_{c.a})\frac{x}{R}, \tag{3}$$

The first term in relation (2) is Plank formula (1) and the second term is a correction for nonzero heat capacity of the frozen body.

The formulas for freezing time obtained by Ryutov [1] and Brazhnikov [3] are different from (2) but give very similar numerical results.

2. Let us have a cylinder with radius R cooled to the cryoscopic temperature T_{cry}. At time $t = 0$, the cylinder is submerged in a cooling agent with temperature T_{cry}; $R - \Delta(t) < r < R$; the temperature of the unfrozen part is assumed to be equal to the cryoscopic temperature T_{cry}. The mathematical formulation of the problem is expressed as

$$\frac{dT}{dt} = \frac{\lambda}{C\rho}\left(\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr}\right);\quad T(R;\, 0) = T_{cry};$$

$$\frac{dT}{dr}(R;\, t) = -\frac{\alpha}{\lambda}(T(R,\, t) - T_{c.a});\quad T(R - \Delta(t);\, t) = T_{cry}; \tag{4}$$

$$q\rho\frac{d\Delta(t)}{dt} = -\lambda\frac{dT}{dr}(R - \Delta(t);\, t);\quad \Delta(0) = 0.$$

Equation (4) is the heat-conduction equation for the frozen part of the body with initial and boundary conditions, the condition that at the freezing front the temperature is equal to the cryoscopic temperature, and the equation of motion of the front with an initial condition.

For convenience, new dimensionless variables are introduced: the radius p, time τ, thickness of the frozen layer δ, and temperature V:

$$p = \frac{r}{R};\quad \tau = \frac{\lambda t}{C\rho R^2};\quad \delta(\tau) = \frac{\Delta}{R};\quad V = \frac{T - T_{c.a}}{T_{cry} - T_{c.a}}.$$

In terms of these variables Eq. (4) looks as follows:

$$\frac{dV}{d\tau} = \frac{d^2V}{dp^2} + \frac{1}{p}\frac{dV}{dp};\quad V(1;\, 0) = 1; \tag{5}$$

$$\frac{dV}{dp}(1;\, \tau) = -\text{Bi}\, V(1;\, \tau); \tag{6}$$

$$V(1 - \delta(\tau);\, \tau) = 1; \tag{7}$$

$$\frac{d\delta(\tau)}{d\tau} = -\frac{1}{\text{Ko}}\frac{dV}{dp}(1 - \delta(\tau);\, \tau);\quad \delta(0) = 0. \tag{8}$$

In order to find an approximate solution of Eq. (5)-(8), the function $V(p, \tau)$ is expressed in a series form as

$$V(p;\, \tau) = \sum_{i=0}^{+\infty} \alpha_i(\tau)\, R_i(p), \tag{9}$$

where the functions $\alpha_i(\tau)$ and $R_i(p)$ are defined as follows:

$$R_0(p) = 1\,;\;\; R_1(p) = \ln p\,;\;\; \alpha_{i+2}(\tau)\,(R''_{i+2}(p) +$$

$$+ (1/p)\,R'_{i+2}(p)) = \alpha'_i(\tau)\,R_i(p)\,;\;\; R_{i+2}(1) = R'_{i+2}(1) = 0\,. \tag{10}$$

In relations (10) the first two terms of the series are a quasisteady approximation, which is subsequently iterated by equation (5). Solving equation (10) for $i = 0, 1$, we obtain

$$R_2(p) = p^2 - 2\ln p - 1\,;\;\; R_3(p) = p^2(\ln p - 1) + \ln p + 1\,;$$

$$\alpha_2(\tau) = \alpha'_0(\tau)/4\,;\;\; \alpha_3(\tau) = \alpha'_1(\tau)/4\,. \tag{11}$$

Substitution of (9) into (6) gives

$$\alpha_1(\tau) = -\,\mathrm{Bi}\,\alpha_0(\tau)\,. \tag{12}$$

Using (9) and (7) and taking only the first two terms of the series, we obtain

$$\alpha_0(\tau) = \frac{1}{1 - \mathrm{Bi}\ln(1-\delta)}\,. \tag{13}$$

Equation (13) relates the surface temperature of the cylinder to the thickness of the frozen layer.

So far, the entire calculation procedure is, in essence, similar to the scheme used in [2] for a plate.

Next, Eq. (9) should be substituted into (8) and only the first four terms taken. Then, we have a differential equation with separated variables with the unknown function $\delta(\tau)$. However, integration of this equation would result in a calculated relation predicting an infinite freezing time. Nevertheless, this obstacle can be overcome. Equation (8) is transformed to the form

$$\delta'(\tau) = \frac{1}{\mathrm{Ko}}\ln p\left(\frac{d}{dp}\left(\frac{V(p;\tau)}{\ln p}\right) + \frac{V(p;\tau)}{p\ln^2 p}\right)\Bigg|_{p=1-\delta(\tau)} =$$

$$= -\frac{1}{\mathrm{Ko}}\left(\ln(1-\delta(\tau))\frac{d}{dp}\left(\frac{V(p;\tau)}{\ln p}\right)\Bigg|_{p=1-\delta(\tau)} + \frac{1}{(1-\delta(\tau))\ln(1-\delta(\tau))}\right)\,. \tag{8'}$$

Substitution of (9) into Eq. (8′) with only the first four terms taken and with allowance for (1)-(13) results in the following equation:

$$\frac{d\tau}{d\delta} = \frac{\mathrm{Ko}}{\mathrm{Bi}}(1 - \mathrm{Bi}\ln(1-\delta))(1-\delta) - \frac{1}{2}(1-\delta) -$$

$$-\frac{2\,\mathrm{Bi}(1-\delta)^2\ln(1-\delta) + (\mathrm{Bi}+1)(2-\delta)}{4(1-\mathrm{Bi}\ln(1-\delta))(1-\delta)\ln(1-\delta)}\,. \tag{14}$$

Equation (14) is easily integrated in quadratures:

$$\tau = \frac{\mathrm{Ko}}{2}\left(\left(1+\frac{2}{\mathrm{Bi}}\right)\left(\delta - \frac{\delta^2}{2}\right) + (1-\delta)^2\ln(1-\delta)\right) - \frac{\delta}{2} + \frac{\delta^2}{4} -$$

$$-\frac{1}{4}\int_0^{\delta}\frac{2\,\mathrm{Bi}(1-x)^2\ln(1-x) + (\mathrm{Bi}+1)\,x\,(2-x)}{(1-\mathrm{Bi}\ln(1-x))(1-x)\ln(1-x)}\,dx\,. \tag{15}$$

Integral (15) cannot be expressed in terms of any tabulated functions. Substituting $\delta = 1$ into (14) and changing the variables $y = -2 \ln (i - x)$ for convenience, we obtain the freezing time. It will be written directly in dimensional quantities:

$$t_f = \frac{R\rho q}{2 (T_{cry} - T_{c.a})} \left(\frac{R}{2\lambda} + \frac{1}{\alpha} \right) + \frac{C\rho R^2}{\lambda} F_1 (\mathrm{Bi}) , \tag{16}$$

where

$$F_1 (\mathrm{Bi}) = \frac{1}{2} \int_0^{+\infty} \frac{\mathrm{Bi} + 1 - (1 + \mathrm{Bi} (1 + y)) \exp (- y)}{y (2 + \mathrm{Bi}\, y)} dy - \frac{1}{4} .$$

The integral $F_1(\mathrm{Bi})$ can only be calculated explicitly at $\mathrm{Bi} \equiv +\infty$ and $\mathrm{Bi} \equiv 0$: $F_1(+\infty) = 0.25$; $F_1(0) = +\infty$. However, $F_1(\mathrm{Bi})$ is easily found numerically. The calculation results show that in the range of the Biot numbers that are usually realized in practice, $\mathrm{Bi} \in (0, 2; 10)$, this integral is well approximated by the expression:

$$F_1 (\mathrm{Bi}) \approx \frac{1}{4} + \frac{0.084 \ln \mathrm{Bi} + 0.27}{\mathrm{Bi}} .$$

Just as in formula (2), in (16) the first term is Plank formula (1) and the second term is a correction for the nonzero heat capacity of the frozen part of the body.

Unfortunately, the temperature distribution that sets in after completion of freezing cannot be calculated, since series (9) diverges at $\tau = \tau_f$.

For the case of a plate, expression (3) is obtained by approximation of $T(x; t_f)$ by a steady-state (i.e., linear, for a plate) distribution, proceeding from known temperatures in the center and on the surface.

In the present case of a cylinder, it is impossible to approximate $T(r; t_f)$ by a steady-state distribution, since this distribution has a singularity in the center. However, the volume-average temperature occurring after completion of freezing can be calculated, and this is the most important point in food technology. To do this, using the known surface temperature (13), we calculate the total heat flux Q through the cylindrical surface for the entire freezing time:

$$Q = \frac{2\pi R^3 (T_{cry} - T_{c.a}) C\rho}{\lambda} \int_0^1 \alpha_0 (\delta) \frac{d\tau}{d\delta} d\delta =$$

$$= \pi R^2 \rho q + \pi R^2 (T_{cry} - T_{c.a}) C\rho \int_0^1 \left(\frac{- \mathrm{Bi} (1 - \delta)}{1 - \mathrm{Bi} \ln (1 - \delta)} - \right.$$

$$\left. - \frac{2 \mathrm{Bi}^2 (1 - \delta) \ln (1 - \delta) + \mathrm{Bi} (\mathrm{Bi} + 1) \delta (2 - \delta)}{2 (1 - \mathrm{Bi} \ln (1 - \delta))^2 (1 - \delta) \ln (1 - \delta)} \right) d\delta .$$

Hence, the volume-average temperature after completion of freezing is equal to

$$T_{av} = T_{c.a} + (T_{cry} - T_{c.a}) F_2 (\mathrm{Bi}) , \tag{17}$$

where

$$F_2 (\mathrm{Bi}) = 1 - \mathrm{Bi} \int_0^{+\infty} \frac{2 (\mathrm{Bi} + 1) - (\mathrm{Bi}\, y^2 + 2 (\mathrm{Bi} + 1) (1 + y)) \exp (- y)}{y (\mathrm{Bi}\, y + 2)^2} dy .$$

Here the integral can be calculated explicitly only for $\mathrm{Bi} = +\infty$ and $\mathrm{Bi} = 0$; $F_2(+\infty) = 0.5$; $F_2(0) = 1$.

Using the numerical results, it is possible to select an approximating formula for $F_2(\mathrm{Bi})$: $F_2(\mathrm{Bi}) \approx (3\mathrm{Bi} + 5)/(6\mathrm{Bi} + 5)$.

3. Let us consider the freezing of a sphere. The notation is the same as in Sect. 2. The heat-conduction equation has the form

$$\frac{dV}{d\tau} = \frac{d^2V}{dp^2} + \frac{2}{p}\frac{dV}{dp}; \quad V(1; 0) = 1 .$$

The other equations of the problem are similar to Eqs. (6)-(8). The function $V(p; \tau)$ is expressed by series (9), in which the functions $\alpha_i(\tau)$ and $R_i(p)$ are defined as

$$R_0(p) = 1; \quad R_1(p) = \frac{1-p}{p};$$

$$\alpha_{i+2}(\tau)\left(R''_{i+2}(p) + \frac{2}{p}R'_{i+2}(p)\right) = \alpha'_i(\tau)\, R_i(p);$$

$$R_{i+2}(1) = R'_{i+2}(1) = 0 .$$

The solution of these equations for $i = 0, 1$ is

$$R_2(p) = p^2 + \frac{2}{p} - 3; \quad R_3(p) = \frac{(1-p)^3}{p};$$

$$\alpha_2(\tau) = \alpha'_0(\tau)/6; \quad \alpha_3(\tau) = -\alpha'_1(\tau)/6 .$$

As in Sect. 2, we obtain

$$\alpha_1(\tau) = \mathrm{Bi}\,\alpha_0(\tau); \quad \alpha_0(\tau) = \frac{1-\delta}{1 + (\mathrm{Bi}-1)\delta}.$$

For a sphere, the same difficulty appears that exists for a cylinder. The same way out of the situation can be suggested. Equation (8) is transformed to

$$\delta'(\tau) = -\frac{1}{\mathrm{Ko}}\frac{1}{p}\left(\frac{d}{dp}(pV(p;\tau)) - V(p;\tau)\right)\Bigg|_{p=1-\delta(\tau)} =$$

$$= -\frac{1}{\mathrm{Ko}}\frac{1}{1-\delta}\left\{\frac{d}{dp}(pV(p;\tau))\Bigg|_{p=1-\delta} - 1\right\}. \tag{8''}$$

Substituting four terms of series (9) into (8''), we obtain the following equation:

$$\frac{d\delta}{d\tau} = \frac{\mathrm{Bi}\,(1 + (\mathrm{Bi}-1)\delta)}{\mathrm{Ko}\,(1-\delta)(1 + (\mathrm{Bi}-1)\delta)^2 + \mathrm{Bi}\,\delta + \frac{1}{2}\mathrm{Bi}\,(\mathrm{Bi}-1)\delta^2}.$$

Integration gives

$$\tau = \frac{\mathrm{Ko}}{\mathrm{Bi}}\left\{-(\mathrm{Bi}-1)\frac{\delta^3}{3} + (\mathrm{Bi}-2)\frac{\delta^2}{2} + \delta\right\} + \frac{\delta^2}{4} +$$

$$+ \frac{\delta}{2(\mathrm{Bi}-1)} - \frac{1}{2(\mathrm{Bi}-1)^2}\ln(1 + (\mathrm{Bi}-1)\delta). \tag{18}$$

Substituting $\delta = 1$ into Eq. (18), we obtain the freezing time. It will be written directly in dimensional quantities:

$$t_f = \frac{R\rho q}{3(T_{cry} - T_{c.a})}\left(\frac{R}{2\lambda} + \frac{1}{\alpha}\right) + \frac{R\rho C}{2}\left\{\frac{R}{2\lambda} + \frac{1}{\alpha}\frac{Bi}{Bi - 1}\left(1 - \frac{\ln Bi}{Bi - 1}\right)\right\}. \quad (19)$$

Just as in the formulas for a plate Eq. (2) and a cylinder Eq. (16), the first term is Plank formula (1) and the second term is a correction for heat capacity of the frozen part.

Just as for a cylinder, in the case of a sphere it is impossible to calculate the temperature distribution upon completion of freezing, but it is possible to determine the volume-average temperature in terms of the total heat flux

$$Q = 4\pi R^2 \alpha (T_{cry} - T_{c.a}) \int_0^{t_f} \alpha_0(\delta)\, d\tau =$$

$$= \frac{4\pi R^4 \alpha (T_{cry} - T_{c.a}) C\rho}{\lambda} \int_0^1 \alpha_0(\delta) \frac{d\tau}{d\delta} d\delta = \frac{4}{3}\pi R^3 q\rho +$$

$$+ \frac{4}{3}\pi R^3 C\rho (T_{cry} - T_{c.a}) \frac{3\,Bi}{2(Bi - 1)}\left\{\frac{1}{2} - \frac{1}{Bi - 1} + \frac{\ln Bi}{(Bi - 1)^2}\right\}.$$

Hence the volume-average temperature of a sphere upon completion of freezing is

$$T_{av} = T_{cry} - (T_{cry} - T_{c.a}) \frac{3\,Bi}{2(Bi - 1)}\left\{\frac{1}{2} - \frac{1}{Bi - 1} + \frac{\ln Bi}{(Bi - 1)^2}\right\}. \quad (20)$$

4. In summary, formulas (2) and (3) (for a plate) are supplemented by formulas (16) and (17) for a cylinder and equations (19) and (20) for a sphere. Thus, the list of simple bodies is exhausted and it is possible to calculate the freezing time and the time necessary for aftercooling for a broad class of foods.

NOTATION

t_f, freezing time, sec; R, radius of cylinder or sphere, m; ρ, density of body, kg/m^3; q, specific heat of phase transition, J/kg; λ, thermal conductivity of frozen part of body, W/(m·K); α, coefficient of heat transfer from surface of body, W(m^2·K); T_{cry}, cryoscopic temperature, K; $T_{c.a}$, temperature of cooling agent, K; C, specific heat of frozen part of body, J/(kg·K); $Bi = \alpha R/\lambda$, Biot number, dimensionless; r, instantaneous radius of cylinder or sphere, m; t, current time, sec; $\Delta(t)$, thickness of frozen layer, m; $T(r; t)$, temperature in frozen part of body, K; $Ko = q/C(T_{cry} - T_{c.a})$, Kossovich number, dimensionless; Q, total heat flux through surface of body for entire freezing time, J; T_{av}, volume-average temperature upon completion of freezing, K.

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